

# Online Appendix to

## *Screening and the Cost of a Prescribed Quantity Schedule*

Dalin Sheng

Southwestern University of Finance and Economics

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This appendix gives extensions and examples of the transition representation, including its incentive-compatible (IC) domain. It also studies fixed stationary targets and duality under moment and monotonicity constraints. Appendix C of the main paper proves Theorem 2. Each class of results is numbered separately and consecutively, with the prefix OA. Equations use a separate consecutive sequence with the same prefix. References without that prefix refer to the main paper.

## A Transition Representation

### A.1 Polynomial transitions

**Proposition OA.1** (Polynomial transition laws). *Let  $\hat{Q}_t = R_0 + \sum_{\ell=1}^N t^\ell A_\ell$ , where  $A_1, \dots, A_N$  are linearly independent signed measures of total mass zero. Let  $Q_t(\cdot \mid z) = \zeta S_0(\cdot \mid z) + (1 - \zeta)\hat{Q}_t(\cdot)$ , where  $\zeta \in [0, 1)$  and  $S_0$  is independent of the action. If  $\mathcal{T}$  contains at least  $N + 1$  distinct action levels, then*

$$d_Q = N, \quad x_N(t) := (t, t^2, \dots, t^N)$$

*is a minimal sufficient statistic. Every other minimal statistic is an invertible affine transformation of  $x_N$ . For any target moment vector  $\bar{m} = (m_1, \dots, m_N)$  whose associated kernel preserves  $\mu$ ,*

$$\mu P_\sigma = \mu \quad \Longleftrightarrow \quad \mathbb{E}_\sigma[T^\ell] = m_\ell, \quad \ell = 1, \dots, N.$$

The same Vandermonde argument gives  $d_Q = J - 1$  for  $J \leq N + 1$  distinct actions. In the preparation-stage technology of Example 1, affine independence of the successor laws is equivalent to linear independence of the polynomial coefficients.

*Proof of Proposition OA.1.* The persistent component cancels across actions:

$$Q_t - Q_{t_0} = (1 - \zeta) \sum_{\ell=1}^N (t^\ell - t_0^\ell) A_\ell.$$

For any  $N + 1$  distinct actions, the corresponding power vectors are affinely independent by the Vandermonde determinant. Linear independence of  $A_1, \dots, A_N$  therefore gives  $d_Q = N$ . Direct integration shows that the power vector  $x_N$  is sufficient.

For minimality, take a continuous sufficient statistic  $y : \mathcal{T} \rightarrow \mathbb{R}^d$  and restrict its codomain to the affine hull of its image. If a finite signed measure  $\omega$  on  $\mathcal{T}$  has total mass zero and  $\int y \, d\omega = 0$ , normalize its positive and negative parts to obtain two lotteries with the same mean statistic. The corresponding constant assignment rules have the same aggregate statistic. Sufficiency and linear independence of the  $A_\ell$ 's imply

$$\int x_N(t) \, d\omega(t) = 0.$$

Choose  $s_0, \dots, s_d$  so that  $y(s_j) - y(s_0)$ ,  $j = 1, \dots, d$ , form a basis. Let  $L$  be the unique affine map with  $L(y(s_j)) = x_N(s_j)$  at these points. Expressing  $y(t) - y(s_0)$  in this basis produces a signed measure with zero total mass and zero  $y$ -moment. The preceding implication gives  $x_N(t) = L(y(t))$ . Thus the power vector is affine in  $y$ . Its affine dimension is  $N$ , so  $d \geq N$  and  $x_N$  is minimal. If  $y$  is also minimal, the linear part of  $L$  is nonsingular, which gives uniqueness up to an invertible affine transformation.

If the kernel associated with  $\bar{m}$  preserves  $\mu$ , then every assignment rule satisfies

$$\mu P_\sigma - \mu = (1 - \zeta) \sum_{\ell=1}^N (\mathbb{E}_\sigma[T^\ell] - m_\ell) A_\ell.$$

Because  $\zeta < 1$  and the coefficient measures are linearly independent, stationarity is equivalent to matching all  $N$  moments. If the action set contains  $J \leq N + 1$  points, the same Vandermonde argument gives affine rank  $J - 1$ . In the independent-stage model, the Bernstein-to-power change of basis gives (2); affine independence of  $R_0, \dots, R_N$  is equivalent to linear independence of the coefficient measures.  $\square$

## A.2 A smooth infinite-rank family

**Example OA.1** (An infinite-dimensional transition family). Let the current and next-period state spaces and the action set all be  $[0, 1]$ . The transition is independent of current type and has density

$$f_t(y) := 1 + \sum_{n=1}^{\infty} 2^{-(n+1)} t^n \cos(2\pi n y)$$

with respect to Lebesgue measure. The series converges uniformly. Each cosine integrates to zero, and  $\sum_{n \geq 1} 2^{-(n+1)} = 1/2$ , so  $f_t$  integrates to one and satisfies  $1/2 \leq f_t(y) \leq 3/2$ . The family is therefore a jointly Feller family of probability kernels and is analytic in  $t$ .

Fix distinct actions  $t_1, \dots, t_N \in (0, 1]$ . If  $\sum_{i=1}^N a_i (Q_{t_i} - Q_0) = 0$ , pairing with  $\cos(2\pi n y)$  gives

$$\sum_{i=1}^N a_i t_i^n = 0, \quad n = 1, \dots, N.$$

The matrix  $(t_i^n)_{n,i=1}^N$  is a Vandermonde matrix with column  $i$  multiplied by  $t_i$ , so it is nonsingular. Hence  $a_1 = \dots = a_N = 0$ . Kernel differences at arbitrarily many actions are linearly independent, so  $d_Q = +\infty$ . A scalar action can therefore generate a smooth transition family with no finite-dimensional sufficient statistic.

## B Necessity on the Incentive-Compatible Domain

This section maintains the notation of Sections 4 and 2 of the main paper.

Theorem 2(i) imposes monotonicity on the conditional mean of  $x_1(T)$ . Incentive compatibility in the screening problem requires the effective allocation to be nondecreasing. Let  $q : \mathcal{Z} \rightarrow (0, \bar{q}]$  be continuous, let  $a := \min \mathcal{T} < b := \max \mathcal{T}$ , and fix  $\beta > 0$  and  $\alpha + \beta a > 0$ . For an assignment rule  $\sigma$ , define

$$r_\sigma(z) := \int_{\mathcal{T}} t \sigma(dt | z), \quad g_\sigma(z) := q(z)[\alpha + \beta r_\sigma(z)].$$

Call  $\sigma$  admissible if  $g_\sigma$  has a nondecreasing representative. A statistic  $x$  is sufficient on the incentive-compatible domain if  $\bar{x}_\sigma = \bar{x}_{\sigma'}$  implies equality of the induced next-period distributions for every pair of admissible assignment rules  $\sigma, \sigma'$ .

**Proposition OA.2** (Necessity on the incentive-compatible domain). *(Proof in Section B.1.)* Maintain the hypotheses of Theorem 2(i). Suppose that

$$\sup_z \frac{\max_{\tilde{z} \leq z} q(\tilde{z})}{q(z)} < \frac{\alpha + \beta b}{\alpha + \beta a}. \quad (\text{OA.1})$$

Suppose also that the mean action is affine in the statistic:

$$t = c_0 + \langle c, x(t) \rangle \quad \text{for every } t \in \mathcal{T} \quad (\text{OA.2})$$

for some  $c_0 \in \mathbb{R}$  and  $c \in \mathbb{R}^d$ . Then  $x$  is sufficient on the incentive-compatible domain if and only if (14) holds. Consequently  $d_Q \leq d$ , and the stationarity conclusion in Theorem 2(iii) applies to a minimal sufficient statistic.

When these conditions hold, fix  $t_0 \in \mathcal{T}$  and define

$$d_{Q,t} := \dim \text{span}\{(t - t_0, \Delta_t) : t \in \mathcal{T}\}.$$

This rank is independent of  $t_0$  and satisfies

$$d_Q \leq d_{Q,t} \leq d, \quad d_{Q,t} \in \{d_Q, d_Q + 1\}.$$

Among continuous statistics that are sufficient on the incentive-compatible domain and satisfy (OA.2), the minimum affine dimension is  $d_{Q,t}$ . Any two minimum-dimensional statistics differ by an invertible affine transformation. In particular,  $d_{Q,t} = d_Q$  if and only if mean action is affine in a minimal sufficient statistic.

The strict range condition (OA.1) permits an interior assignment that can be perturbed in both directions while retaining IC. Under the affine mean condition, lotteries with the same statistic also have the same mean. Substituting one for another therefore preserves the effective allocation. The rank  $d_{Q,t}$  counts the coordinates needed to represent both the transition law and mean action.

**Example OA.2** (The mean action may require an additional coordinate). Let  $\mathcal{T} = [0, 1]$ , define

$$s(t) := 4 \left( t - \frac{1}{2} \right)^2,$$

and set

$$Q_t = [1 - s(t)]\delta_0 + s(t)\delta_1.$$

Because  $s(t) \in [0, 1]$ , this is a well-defined transition family. For any reference action  $t_0$ ,

$$Q_t - Q_{t_0} = [s(t) - s(t_0)](\delta_1 - \delta_0).$$

Hence  $d_Q = 1$ , and  $s$  is a minimal sufficient statistic. Mean action is not determined by this statistic, since  $s(0) = s(1) = 1$ . By Theorem 2(ii), every one-dimensional minimal statistic is an affine transformation of  $s$ , so none determines the mean.

Take  $t_0 = 1/2$  and write  $D := \delta_1 - \delta_0$ . The augmented vectors

$$\left( -\frac{1}{2}, D \right) \quad \text{and} \quad \left( \frac{1}{2}, D \right)$$

are linearly independent. Therefore

$$d_{Q,t} = 2 = d_Q + 1.$$

The statistic  $(t, s(t))$  is sufficient and records mean action, while no one-dimensional statistic can do both. With a constant quantity schedule, the strict range condition holds, so the same dimension gap arises on the incentive-compatible domain.

## B.1 Proof of Proposition OA.2

*Proof of Proposition OA.2.* Integration proves sufficiency of (14). For necessity, write  $h = 1/q$  and  $g_\sigma = q(\alpha + \beta r_\sigma)$ .

*An interior admissible baseline.* Define

$$g_0(z) := (\alpha + \beta a) \max_{\tilde{z} \leq z} q(\tilde{z}).$$

The running maximum of a continuous function is continuous, so  $g_0$  is continuous and nondecreasing. It also satisfies  $g_0 \geq (\alpha + \beta a)q$ . By (OA.1), the continuous function  $(\alpha + \beta b)q - g_0$  is strictly positive

on  $\mathcal{Z}$ . Let  $\varepsilon_0 > 0$  be a lower bound. Choose  $\kappa, c > 0$  such that  $\kappa + c(\bar{z} - \underline{z}) < \varepsilon_0$ , and set

$$g^b(z) := g_0(z) + \kappa + c(z - \underline{z}).$$

Then

$$g^b(z') - g^b(z) \geq c(z' - z) \quad (z < z'),$$

and

$$(\alpha + \beta a)q(z) + \kappa \leq g^b(z) \leq (\alpha + \beta b)q(z) - \varepsilon_1, \quad \varepsilon_1 := \varepsilon_0 - \kappa - c(\bar{z} - \underline{z}) > 0.$$

Thus the associated conditional mean

$$r^b(z) := \frac{g^b(z)/q(z) - \alpha}{\beta}$$

is uniformly bounded away from  $a$  and  $b$ . In particular, with  $\eta_0 := \min\{\kappa, \varepsilon_1\}/(\beta\bar{q}) > 0$ ,

$$a + \eta_0 \leq r^b(z) \leq b - \eta_0 \quad \text{for every } z.$$

Fix actions  $t, u \in \mathcal{T}$ . Choose a lottery  $\pi^*$  that assigns positive probability to every point of a finite set containing  $t$  and  $u$ , and let  $\bar{t}^* := \int s \pi^*(ds)$ . For sufficiently small  $\gamma_b \in (0, 1)$ ,

$$\frac{r^b(z) - \gamma_b \bar{t}^*}{1 - \gamma_b} \in [a, b] \quad \text{for every } z.$$

The bracketing construction in Lemma 1 provides a Borel family of lotteries  $\pi_z^b$  with these means. Define

$$\sigma^b(dv | z) := (1 - \gamma_b)\pi_z^b(dv) + \gamma_b\pi^*(dv).$$

Then  $r_{\sigma^b} = r^b$ , so  $g_{\sigma^b} = g^b$ . The assignment is admissible, and it places a uniform positive mass on every action in the support of  $\pi^*$ .

*Type-dependent perturbations.* Let  $f \in C^1(\mathcal{Z})$  satisfy  $\mathbb{E}[hf] = 0$ , and define  $\tilde{f} := hf$ . For sufficiently small  $\varepsilon > 0$ , set

$$\sigma^\pm(dv | z) := \sigma^b(dv | z) \pm \varepsilon \tilde{f}(z)[\delta_t - \delta_u](dv).$$

The baseline places uniformly positive mass on the perturbed actions, so these kernels remain nonnegative. Their conditional means are  $r^b \pm \varepsilon \tilde{f}(t - u)$ , and hence

$$g_{\sigma^\pm}(z) = g^b(z) \pm \varepsilon \beta(t - u)f(z).$$

For  $z < z'$ ,

$$g_{\sigma^\pm}(z') - g_{\sigma^\pm}(z) \geq [c - \varepsilon \beta |t - u| \|f'\|_\infty](z' - z).$$

After reducing  $\varepsilon$  if necessary, both assignment rules are admissible and their conditional means

remain in  $[a, b]$ . They also have the same aggregate statistic because

$$\bar{x}_{\sigma^\pm} = \bar{x}_{\sigma^b} \pm \varepsilon \mathbb{E}[\tilde{f}][x(t) - x(u)] = \bar{x}_{\sigma^b}.$$

Sufficiency on the incentive-compatible domain gives, for every  $\phi \in C(\mathcal{Z})$ ,

$$\int h(z)f(z)\{Q_t\phi(z) - Q_u\phi(z)\}\mu(dz) = 0.$$

The functions  $hf$  with  $f \in C^1(\mathcal{Z})$  and  $\mathbb{E}[hf] = 0$  are dense in the zero-mean subspace of  $L^2(\mu)$ . To see this, take a zero-mean  $u_0 \in L^2(\mu)$  and choose  $f_n \in C^1(\mathcal{Z})$  with  $hf_n \rightarrow u_0$  in  $L^2$ . Set  $c_n := \mathbb{E}[hf_n]/\mathbb{E}[h]$  and  $\hat{f}_n := f_n - c_n$ . Then  $\mathbb{E}[h\hat{f}_n] = 0$  and  $h\hat{f}_n \rightarrow u_0$ . The displayed orthogonality therefore implies that  $Q_t\phi - Q_u\phi$  is constant  $\mu$ -almost everywhere. Joint Feller continuity and full support make it constant everywhere. Continuous functions separate finite measures, so the kernel difference

$$Q_t(\cdot \mid z) - Q_u(\cdot \mid z)$$

is independent of  $z$ .

*Affine factorization on the admissible domain.* Fix  $t_0$  and write  $\Delta_t := Q_t - Q_{t_0}$ . Choose  $t_1, \dots, t_d$  so that  $x(t_j) - x(t_0)$  is a basis of  $\mathbb{R}^d$ , and let  $D_1, \dots, D_d$  solve (C.1). For an arbitrary action  $t$ , choose coefficients  $c_j$  satisfying

$$x(t) - x(t_0) = \sum_{j=1}^d c_j [x(t_j) - x(t_0)],$$

and define

$$\omega := \delta_t - \delta_{t_0} - \sum_{j=1}^d c_j (\delta_{t_j} - \delta_{t_0}).$$

This signed measure has total mass zero and zero  $x$ -moment. By (OA.2), it also has zero mean action. If  $\omega \neq 0$ , let  $M$  be the common mass of its positive and negative parts, and define

$$\pi_+ := \frac{\omega^+}{M}, \quad \pi_- := \frac{\omega^-}{M}, \quad \bar{\pi} := \frac{\pi_+ + \pi_-}{2}.$$

Use the interior baseline above, with  $\pi^*$  chosen to charge the support of  $\omega$ , and set

$$\sigma_\pm(\cdot \mid z) := \sigma^b(\cdot \mid z) + \gamma'[\pi_\pm - \bar{\pi}]$$

for sufficiently small  $\gamma' > 0$ . The two assignments are well defined. They have the same conditional mean, namely  $r^b$ , and hence the same effective allocation  $g^b$ . They are therefore admissible. Their

aggregate statistics also coincide. Since the kernel differences are independent of type,

$$\mu P_{\sigma_+} - \mu P_{\sigma_-} = \frac{\gamma'}{M} \int_{\mathcal{T}} \Delta_v \, d\omega(v).$$

Sufficiency forces the right-hand side to vanish. Hence  $\Delta_t = \sum_j c_j \Delta_{t_j}$ . When  $\omega = 0$ , the same conclusion holds directly. Together with (C.1), this proves (14).

The stationarity statement in Theorem 2(iii) depends only on this representation and on linear independence of the representing measures. It therefore continues to hold. The same representation gives  $d_Q \leq d$ .

*The augmented dimension.* For each action, define

$$F_{t_0}(t) := (t - t_0, \Delta_t) \in \mathbb{R} \times \mathcal{D}_Q.$$

If  $t_1$  is another reference action, then

$$F_{t_1}(t) = F_{t_0}(t) - F_{t_0}(t_1), \quad F_{t_1}(t_0) = -F_{t_0}(t_1).$$

The spans generated by the two reference curves are therefore identical, so  $d_{Q,t}$  is independent of the reference action. Projection of  $\text{span } F_{t_0}(\mathcal{T})$  onto its measure component has image  $\mathcal{D}_Q$  and a kernel of dimension at most one. Hence

$$d_Q \leq d_{Q,t} \leq d_Q + 1.$$

Now take any sufficient statistic that also satisfies (OA.2), and remove redundant coordinates. If its affine dimension is  $d$ , then

$$F_{t_0}(t) = \left( \langle c, x(t) - x(t_0) \rangle, \sum_{i=1}^d [x_i(t) - x_i(t_0)] D_i \right).$$

Thus  $F_{t_0}$  factors linearly through  $x(t) - x(t_0)$ , and  $d_{Q,t} \leq d$ .

Conversely, choose a basis  $E_1, \dots, E_{d_{Q,t}}$  of  $\text{span } F_{t_0}(\mathcal{T})$ , and write

$$F_{t_0}(t) = \sum_{i=1}^{d_{Q,t}} y_i(t) E_i, \quad y(t_0) = 0.$$

The coordinate map  $y$  is continuous. Pair the measure components of the basis vectors with finitely many continuous functions separating their span, and include the scalar component. This gives a nonsingular coordinate matrix. The joint Feller property makes the paired coordinates of  $F_{t_0}(t)$  continuous. In this representation, the scalar components express  $t$  as an affine function of  $y(t)$ , and the measure components give an affine kernel representation. Thus  $y$  is sufficient and satisfies (OA.2). Since  $F_{t_0}(\mathcal{T})$  spans the augmented space,  $\text{aff } y(\mathcal{T}) = \mathbb{R}^{d_{Q,t}}$ . The lower bound is attained.

A minimum-dimensional statistic therefore gives a linear isomorphism from its coordinate space to  $\text{span } F_{t_0}(\mathcal{T})$ . Two such statistics are coordinate systems for two bases of the same space, so they differ by an invertible affine transformation. Finally,  $d_{Q,t} = d_Q$  exactly when the scalar coordinate  $t$  is affine in a minimal sufficient statistic.  $\square$

## C Identifying a Fixed Stationary Target

The dimension in Theorem 2 measures how many statistics are needed to recover every successor distribution. To test whether an assignment reproduces one fixed target, fewer equalities can suffice. This reduction depends on the target's position in the attainable set.

### C.1 Identifying a fixed target

Fix a minimal sufficient statistic  $x : \mathcal{T} \rightarrow \mathbb{R}^{d_Q}$ , and let

$$\mathcal{X} := \text{co } x(\mathcal{T}).$$

Take  $\bar{x} \in \mathcal{X}$  such that  $\mu Q_{\bar{x}} = \mu$ . By Theorem 2(iii), the stationary assignment rules are

$$\mathfrak{S}(\bar{x}) := \{\sigma : \mu P_{\sigma} = \mu\} = \{\sigma : \bar{x}_{\sigma} = \bar{x}\}.$$

A linear map  $\mathbf{H} : \mathbb{R}^{d_Q} \rightarrow \mathbb{R}^r$  identifies the target  $\bar{x}$  on the attainable set if

$$\mathfrak{S}(\bar{x}) = \{\sigma : \mathbf{H}\bar{x}_{\sigma} = \mathbf{H}\bar{x}\}.$$

Equivalently,

$$\mathcal{X} \cap (\bar{x} + \ker \mathbf{H}) = \{\bar{x}\}.$$

Define the minimum rank needed to identify the target by

$$r_{\text{id}}(\bar{x}) := \min\{\text{rank } \mathbf{H} : \mathcal{X} \cap (\bar{x} + \ker \mathbf{H}) = \{\bar{x}\}\}.$$

Set  $r_{\text{id}}(\bar{x}) = 0$  when  $d_Q = 0$ . If  $d_Q > 0$ , then

$$1 \leq r_{\text{id}}(\bar{x}) \leq d_Q.$$

An interior target requires all  $d_Q$  coordinates:

$$r_{\text{id}}(\bar{x}) = d_Q \quad \text{when } \bar{x} \in \text{ri } \mathcal{X}.$$

If the target is an exposed point of  $\mathcal{X}$ , one exposing linear functional identifies it, so  $r_{\text{id}}(\bar{x}) = 1$ .

## C.2 A quadratic moment example

For a quadratic transition law with  $\mathcal{T} = [0, 1]$ , the attainable moment set is

$$\mathcal{X}_2 = \{(m, k) : 0 \leq m \leq 1, m^2 \leq k \leq m\}.$$

An interior target requires two independent affine restrictions. For a deterministic target  $(t_0, t_0^2)$ , the single equality

$$\mathbb{E}[(T - t_0)^2] = \mathbb{E}[T^2] - 2t_0\mathbb{E}[T] + t_0^2 = 0$$

forces  $T = t_0$  almost surely and identifies the target. Recovering all attainable successor distributions still requires the two-dimensional statistic.

## D Duality for Stationary Allocation

Duality expresses the allocation problem through prices on aggregate moments and a multiplier on the monotonicity constraint.

Let  $L^\infty_\uparrow(\mu)$  denote the cone of essentially bounded functions that admit a nondecreasing representative. Its polar cone is

$$\mathcal{H}_\mu := \left\{ \xi \in L^1(\mu) : \mathbb{E}[\xi] = 0, \quad X_\xi(x) := \int_{[\underline{z}, x]} \xi(z) \mu(dz) \geq 0 \text{ for every } x \right\}.$$

Each  $\xi \in \mathcal{H}_\mu$  has nonpositive pairing with every nondecreasing allocation and serves as a multiplier on the incentive constraint.

For a Carathéodory integrand  $\mathbf{c} : \mathcal{Z} \times \mathcal{T} \rightarrow \mathbb{R}$  satisfying

$$\int_{\mathcal{Z}} \sup_{t \in \mathcal{T}} |\mathbf{c}(z, t)| \mu(dz) < \infty,$$

let  $C_{\bar{m}}(\mathbf{c})$ , where  $\bar{m} = (m_1, \dots, m_N)$ , be the minimum of  $\mathbb{E}[\mathbf{c}(Z, T)]$  over assignment rules that implement  $q$  and preserve  $\mu$ . After removing the constant term  $\alpha \mathbb{E}[Zq(Z)]$  and dividing by  $\beta$ , maximizing current output is equivalent to minimizing  $\mathbf{c}(z, t) = -w(z)t$ .

**Theorem OA.1** (Implementation, stationarity, and duality). *Maintain Assumption 1. Let  $\bar{m} \in \mathcal{M}_N(\mathcal{T})$  be the unique moment vector in (5).*

(i) *An assignment rule can be paired with payments to implement  $q$  while preserving  $\mu$  if and only if*

$$g_\sigma \in L^\infty_\uparrow(\mu), \quad \mathbb{E}[T^\ell] = m_\ell \quad (\ell = 1, \dots, N). \quad (\text{OA.3})$$

*Any assignment rule satisfying these conditions almost everywhere can be modified on a  $\mu$ -null set to obtain a pointwise direct mechanism. For any pointwise version, truthful utility and*

payments are

$$V(z) = V_0 + \int_{\underline{z}}^z g_\sigma(x) dx, \quad (\text{OA.4})$$

$$p(z) = zg_\sigma(z) - V(z), \quad (\text{OA.5})$$

and individual rationality is equivalent to  $V_0 \geq 0$ .

(ii) If (OA.3) is feasible, then

$$C_{\bar{m}}(\mathbf{c}) = \sup_{\substack{\varpi \in \mathbb{R}^N \\ \xi \in \mathcal{H}_\mu}} \left\{ - \sum_{\ell'=1}^N \varpi_{\ell'} m_{\ell'} + \int \left[ \alpha q(z) \xi(z) + \min_{t \in \mathcal{T}} \left\{ \mathbf{c}(z, t) + [\varpi_1 + \beta q(z) \xi(z)] t + \sum_{\ell'=2}^N \varpi_{\ell'} t^{\ell'} \right\} \right] \mu(dz) \right\}. \quad (\text{OA.6})$$

There is no duality gap.

(iii) For the normalized output problem, let  $\mathbf{c}(z, t) = -w(z)t$ . If a dual maximizer  $(\varpi^*, \xi^*)$  exists, every primal optimizer  $\sigma^*$  satisfies, for  $\mu$ -almost every  $z$ ,

$$\sigma^* \left( \arg \min_{s \in \mathcal{T}} \left\{ \sum_{\ell'=2}^N \varpi_{\ell'}^* s^{\ell'} + [\varpi_1^* - w(z) + \beta q(z) \xi^*(z)] s \right\} \middle| z \right) = 1. \quad (\text{OA.7})$$

The vector  $\varpi$  prices the moment restrictions, and  $\xi$  adjusts the linear coefficient in the action problem to account for monotonicity. Equation (OA.6) equates the primal minimum with the dual supremum. If that supremum is attained, (OA.7) characterizes optimal lottery support. For  $B(\bar{m}) := \max \mathbb{E}[wT]$ , an attained multiplier  $\varpi_\ell$  is a supergradient with respect to  $m_\ell$ ; in units of output, the multiplier is  $\beta \varpi_\ell$ .

**Generalized moment restrictions.** The same duality applies to finitely many generalized moment restrictions

$$\mathbb{E}[\varphi_j(Z, T)] = c_j, \quad j = 1, \dots, J,$$

where each  $\varphi_j$  is Carathéodory and satisfies  $\int \sup_t |\varphi_j(z, t)| \mu(dz) < \infty$ . The minimax argument gives the pointwise objective

$$\mathbf{c}(z, t) + \sum_j \varpi_j \varphi_j(z, t) + \beta q(z) \xi(z) t.$$

Joint moments of  $(Z, T)$  accommodate type-dependent transition effects. For a separable transition law, any sufficient statistic  $x$  can express the restrictions. A minimal statistic gives  $d_Q$  aggregate multipliers.

## D.1 Proof of the duality theorem

*Proof of Theorem OA.1. Implementation and stationarity.* Let  $V$  denote truthful utility. For  $z > z_0$ , incentive compatibility implies

$$V(z_0) + (z - z_0)g_\sigma(z_0) \leq V(z) \leq V(z_0) + (z - z_0)g_\sigma(z).$$

These inequalities imply that  $g_\sigma$  is nondecreasing and, by Darboux sums,

$$V(z) - V(z_0) = \int_{z_0}^z g_\sigma(x) \, dx.$$

The same inequalities prove the converse. Substitution gives (OA.4) and (OA.5). Since  $g_\sigma > 0$ , individual rationality is equivalent to  $V_0 \geq 0$ .

For an effective allocation defined almost everywhere, take right limits at interior points and a left limit at the upper endpoint. This gives a nondecreasing Borel version. Full support makes the complement of the exceptional null set dense. Passing to limits through that set preserves the continuous bounds  $q(\alpha + \beta a)$  and  $q(\alpha + \beta b)$  at every report. The implied conditional mean lies in  $[a, b]$ . On the exceptional set, implement that mean with a Borel endpoint lottery. This modification leaves all aggregate moments unchanged.

Integrating (3) and subtracting (5) gives

$$\mu P_\sigma - \mu = (1 - \zeta) \sum_{\ell=1}^N (\mathbb{E}[T^\ell] - m_\ell) A_\ell.$$

Since  $\zeta < 1$  and the  $A_\ell$ 's are linearly independent, stationarity is equivalent to (6). This proves part (i).

*The monotonicity cone.* The polar of the cone of nondecreasing  $L^\infty(\mu)$  classes, in the dual pair  $(L^\infty, L^1)$ , is  $\mathcal{H}_\mu$ . Constants imply  $\mathbb{E}[\xi] = 0$ , upper-interval indicators imply  $X_\xi \geq 0$ , and Stieltjes integration gives, for every nondecreasing  $s$ ,

$$\int \xi s \, d\mu = - \int_{(\underline{z}, \bar{z}]} X_\xi(x) \, ds(x) \leq 0.$$

For the converse, suppose the set of pairs  $(z, z')$  with  $z < z'$  and  $g(z) > g(z')$  has zero  $\mu \otimes \mu$ -measure. By Fubini's theorem, for almost every  $z$ ,

$$g(y) \leq g(z) \quad \text{for almost every } y < z, \quad g(z) \leq g(y) \quad \text{for almost every } y > z.$$

For a rational point  $r$  in the state interval, define

$$a(r) := \operatorname{ess\,sup}_{y < r} g(y), \quad \tilde{g}(z) := \inf_{r > z, \, r \in \mathbb{Q}} a(r),$$

with the natural endpoint conventions. The function  $\tilde{g}$  is nondecreasing. At every  $z$  with the Fubini property, its left limit is at most  $g(z)$ . Full support gives  $a(r) \geq g(z)$  for every rational  $r > z$ , so  $g(z) \leq \tilde{g}(z)$ . At each continuity point of  $\tilde{g}$  with the Fubini property, the two inequalities imply  $g(z) = \tilde{g}(z)$ . A monotone function has only countably many discontinuities, and  $\mu$  is atomless. Thus  $\tilde{g}$  is a nondecreasing representative of  $g$ .

If  $g$  has no nondecreasing representative, the violation set must therefore have positive  $\mu \otimes \mu$ -measure. Decomposing it over rational separating points and rational levels yields numbers  $c_- < c_+$  and points  $x < y$  such that

$$A := \{g \geq c_+\} \cap [\underline{z}, x], \quad B := \{g \leq c_-\} \cap [y, \bar{z}]$$

have positive measure. Define

$$\xi := \frac{\mathbf{1}_A}{\mu(A)} - \frac{\mathbf{1}_B}{\mu(B)}.$$

Then  $\xi \in \mathcal{H}_\mu$ , while  $\langle \xi, g \rangle \geq c_+ - c_- > 0$ . Hence

$$L^\infty_\uparrow(\mu) = \{g \in L^\infty(\mu) : \langle \xi, g \rangle \leq 0 \text{ for every } \xi \in \mathcal{H}_\mu\}. \quad (\text{OA.8})$$

This representation also shows that the monotonicity cone is weak-star closed.

*Duality.* Represent an assignment rule by the Young measure

$$\gamma(\mathrm{d}z, \mathrm{d}t) = \mu(\mathrm{d}z)\sigma(\mathrm{d}t \mid z).$$

The set  $\Gamma$  of such measures is compact and convex in the stable topology generated by  $L^1(\mu; C(\mathcal{T}))$  (Balder, 2000). For  $y = (\varpi, \xi) \in \mathbb{R}^N \times \mathcal{H}_\mu$ , write the Lagrangian as

$$\begin{aligned} \mathcal{L}(\gamma; y) = & - \sum_{\ell'=1}^N \varpi_{\ell'} m_{\ell'} \\ & + \int_{\mathcal{Z} \times \mathcal{T}} \left\{ \mathfrak{c}(z, t) + \sum_{\ell'=1}^N \varpi_{\ell'} t^{\ell'} + \xi(z) q(z) [\alpha + \beta t] \right\} \gamma(\mathrm{d}z, \mathrm{d}t). \end{aligned}$$

The expression is independent of the chosen conditional-mean version. Its integrand belongs to  $L^1(\mu; C(\mathcal{T}))$ , so the Lagrangian is continuous in  $\gamma$  under the stable topology.

By (OA.8),

$$\sup_{\xi \in \mathcal{H}_\mu} \langle \xi, g \rangle = \begin{cases} 0, & g \in L^\infty_\uparrow(\mu), \\ +\infty, & g \notin L^\infty_\uparrow(\mu). \end{cases}$$

The unrestricted multipliers  $\varpi$  impose the moment equalities in the same way. Thus the supremum over  $y$  equals the objective on feasible assignment rules and is  $+\infty$  otherwise.

Equip  $\mathbb{R}^N \times \mathcal{H}_\mu$  with the product of the Euclidean and  $L^1(\mu)$ -norm topologies. For fixed  $y$ , the Lagrangian is continuous and affine on the compact convex set  $\Gamma$ . For fixed  $\gamma$ , it is continuous and

affine in  $y$ , since the effective allocation is bounded. Sion's minimax theorem (Sion, 1958) allows the minimum and supremum to be interchanged. Measurable selection (Aliprantis and Border, 2006, Chapter 18) then reduces the inner minimum to a pointwise choice over compact  $\mathcal{T}$ . This gives (OA.6).

Suppose both primal and dual optima are attained. For a feasible primal optimizer  $\gamma^*$ , the moment terms vanish and the polar pairing is nonpositive, so  $\mathcal{L}(\gamma^*; y^*) \leq C_{\bar{m}}(\mathfrak{c})$ . Dual optimality gives the reverse inequality. Hence the polar pairing is zero, and the integral of the nonnegative pointwise minimization gap also vanishes. With  $\mathfrak{c} = -wt$ , this yields (OA.7) and proves part (iii).  $\square$

## References

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